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ANALYSIS OF A WIRE IN A RECTANGULAR CAVITY

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Abstract

An efficient method of analyzing a loaded wire enclosed within a rectangular cavity is developed. The wire and the cavity interior are excited by electromagnetic sources exterior to the cavity. The formulation of the problem makes use of the theory of Fourier series expansion to approximate the waveform of unknown currents excited on the wire. This method bypasses the dyadic Green's function approach thereby leading to a solution which is computationally easier to handle.

Introduction

The dyadic Green's function is a powerful tool in mathematical Physics and has been extensively used in the analysis of a rectangular cavity. Tai and Rozenfeld[1] gave a detailed derivation of several different and equivalent representations of the dyadic Green's functions for a rectangular cavity. Rahmat-Samii[2] derived the Green's functions for rectangular cavities and waveguides by using the theory of distribution. However, they did not consider the problem of interaction with a wire within a cavity. Recently, Seidel[3] has considered the problem of determining currents excited on a wire in a rectangular cavity. He has formulated an integral equation by using the Green's function approach and then solved numerically by the method of moments. The dyadic Green's functions for a problem of this nature are difficult to compute numerically. Seidel gave an extensive numerical analysis to arrive at a comparatively easier solution. Still, the dilemma of large computer time and computational complexity in case of Green's function approach warrants further work in this area.

This paper presents an efficient approach that treats the problem with a minimum of numerical efforts. The wire current is represented by a truncated Fourier series expansion with unknown Fourier coefficients which are determined by appropriately enforcing boundary conditions. Different load conditions on the wire are easily treated in this method.

Formulation of the Problem

The rectangular cavity under consideration has the configuration shown in Figure 1. The dimensions of the cavity are represented by a, b and c in the X, Y and Z directions respectively of the Cartesian coordinate system. For purposes of this problem, consider a perfectly conducting, round, thin wire of radius r ($r << \lambda$) running parallel to the X-axis into the cavity interior through a small hole located at $r_a = (a, b', c')$. The wire is excited by electromagnetic sources exterior to the cavity. We assume further that the media in both the interior and exterior regions are linear, homogeneous, isotropic and lossless, and are characterized by free-space parameters (μ_0, ϵ_0). However, the extension to lossy media is effected by replacing ϵ_0 by a complex permittivity $\hat{\epsilon}$. The time harmonic variation with angular frequency ω and the factor $e^{j\omega t}$ are understood throughout.

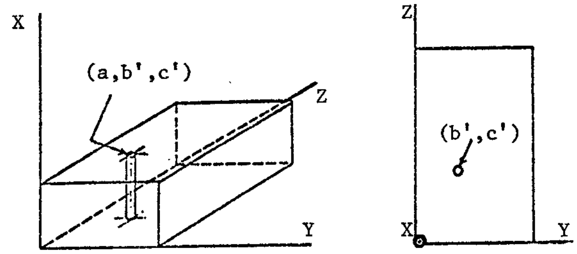

Fig. 1 Configuration of a rectangular cavity

Consider the time-harmonic complex Helmholtz equation

$$\nabla^2\underline{A} + k^2\underline{A} = -\underline{J} \tag{1}$$

where $\underline{A}$ is the magnetic vector potential, k is the wave number of the medium of the cavity interior, and $\underline{J}$ is the source or impressed current.

Let us assume that the current on the thin wire inside the cavity is x-directed and that it undergoes variation with respect to x. In view of this, the unknown current on the wire is represented by a Fourier cosine series in the interval $0 < x < a$ as

$$I(x) = \sum_{v=0}^{\infty} B_v \cos\frac{v\pi}{a}x \tag{2}$$

where B's are the Fourier coefficients yet to be determined. The representation of Eq. (2) converges to I(x) on the closed interval $0 \le x \le a$.

Thus, from Eq. (2) we have

$$\underline{J} = \underline{U}_x \sum_{v=0}^{\infty} B_v \cos\frac{v\pi}{a}x\,\delta(y-b')\delta(z-c') \tag{3}$$

where $\underline{U}_x$ is the unit vector in the direction of X and δ is the impulse function.

Note that $\underline{J}$ is x-directed and the wire is thin; so it is expected that the x-directed $\underline{A}$ is sufficient for representing the field inside the cavity. Thus the Eq. (1) reduces to

$$(\nabla^2+k^2)A_x = -\sum_{v=0}^{\infty} B_v \cos\frac{v\pi}{a}x\,\delta(y-b')\delta(z-c') \tag{4}$$

In terms of the magnetic vector potential we can express the electric field $\underline{E}$ as

$$\underline{E} = -j\omega\mu_0\underline{A} + \frac{1}{j\omega\epsilon_0}\nabla(\nabla\cdot\underline{A}) \tag{5}$$

where μ_0 and ϵ_0 are the free space permeability and permittivity respectively. An x-component $\underline{E}$ is readily written from Eq. (5) as

$$E_x = \frac{1}{j\omega\epsilon_0}\left(\frac{\partial^2}{\partial x^2} + k^2\right)A_x \tag{6}$$

Eliminating A_x from Eq. (4) by Eq. (6) we have

$$(\nabla^2+k^2)E_x = \frac{j}{\omega\epsilon_0}\delta(y-b')\delta(z-c')\sum_{v=0}^{\infty}\cdot$$

$$\left\{k^2-\left(\frac{v\pi}{a}\right)^2\right\} B_v\cos\frac{v\pi}{a}x \tag{7}$$

The general expression for the x-polarized electric field $\underline{E}$ inside the cavity is

$$E_x = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{mnp} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \sin\frac{p\pi}{c}z \quad (8)$$

where A's are unknown coefficients.

Obviously, the field represented by Eq. (8) satisfies the boundary conditions on the walls of the cavity. However, it remains to satisfy the boundary condition on the surface of the wire. If we substitute the value of E_x from Eq. (8) into Eq. (7) and perform integrations on both sides by forming suitable inner products, we have

$$A_{mnp} = \frac{j4B_m\{k^2-(\frac{m\pi}{a})^2\}}{bc(k^2-K_{mnp}^2)\omega\epsilon_0} \sin\frac{n\pi}{b} b' \sin\frac{p\pi}{c} c' \quad (9)$$

where

$$K_{mnp}^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 + (\frac{p\pi}{c})^2 \quad (10)$$

In order to evaluate the coefficients B's, we consider the boundary condition on the surface of the wire which is given by

$$\hat{n} \times \underline{E}_{tot} = \hat{z}(x)I(x) \quad (11)$$

where $\hat{n}$ is the unit normal vector on the surface of the wire, $\underline{E}_{tot}$ is the total electric field and $\hat{z}(x)$ is the impedance function of position. In view of the assumptions made earlier, Eq. (11) reduces to

$$E_x - \hat{z}(x)I(x) = -E_{tan}^i \quad (12)$$

where E_{tan}^i is the tangential component of the impressed electric field.

Let us define a function of the following form

$$W_u = \cos\frac{u\pi}{a}x \,\delta(y-b')\delta(z-c') \quad (13)$$

Substituting I(x) and E(x) from Eqs. (2) and (8) into Eq. (12), multiplying both sides of the resulting equation scalarly by W_u and finally performing integrations within the limits of cavity dimensions one obtains

$$\sum_{v=0}^{M} Z_{uv}B_v + \sum_{v=0}^{M} ZL_{uv}B_v = V_u, \; u=0,1,2,..,M \quad (14)$$

where

$$Z_{uv} = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{4\{k^2-(\frac{v\pi}{a})^2\}\sin^2\frac{n\pi}{b}b'\sin^2\frac{p\pi}{c}c'}{jbc(k^2-K_{vnp}^2)\omega\epsilon_0} \int_0^a \cos\frac{u\pi}{a}x \cos\frac{v\pi}{a}x \, dx \quad (15)$$

$$ZL_{uv} = \int_0^a \hat{z}(x)\cos\frac{u\pi}{a}x \cos\frac{v\pi}{a}x \, dx \quad (16)$$

$$V_u = \int_0^a E_{tan}^i \cos\frac{u\pi}{a}x \, dx \quad (17)$$

Note that we have truncated the Fourier series. In a matrix form Eq. (14) is

$$[Z][B] + [ZL][B] = [V] \quad (18)$$

Finally

$$[B] = [Z + ZL]^{-1}[V] \quad (19)$$

This completes the solution to the problem of determining currents excited on the wire in a rectangular cavity.

Note in Eq. (15) that $Z_{uv} = 0$ if $u \neq v$. Finally, Eq. (15) reduces to

$$Z_{uu} = \frac{ja\{k^2-(\frac{u\pi}{a})^2\}}{b\epsilon_u\omega\epsilon_0} \sum_{n=1}^{\infty} \cdot \frac{\cos h(\alpha c) - \cos h(\alpha(2c'-c))}{\alpha \sin h(\alpha c)} \sin^2\frac{n\pi}{b}b' \quad (20)$$

where

$$\alpha^2 = (\frac{u\pi}{a})^2 + (\frac{n\pi}{b})^2 - (\frac{2\pi}{\lambda})^2 \quad (21)$$

and $\epsilon_u = 1$ for $u = 0$ and $\epsilon_u = 2$ for $u \neq 0$.

Numerical Results

Selected numerical results are presented to describe the electromagnetic behavior of a rectangular cavity having a wire within it. Results are computed for a 3×4×5 cavity with the wire being located at (b', c') = (1·5, 2). All units are in MKS system. We have chosen $\hat{z}(x) = Z_L\delta(x)$ and $E_{tan}^i = \delta(x-3)$, where Z_L is the terminating load.

Figure 2 shows currents along the wire for different number of Fourier coefficients. It is observed that only the first 4 coefficients are good enough for obtaining convergence. Figure 3, shows the convergence behavior of the current distribution for different values of mode index p with respect to the largest dimension of the cavity. It is observed that the convergence to the desired result is achieved for p = 5000. This number, though large, does not pose any serious problem with regard to the computing cost. However, it is observed that the maximum error with p = 1000 is of the order of 10% which is still good enough to work with. Figure 4 shows the current variation for different frequencies. The effects of various load conditions are shown in figure 5. The current goes to zero, as asserted, when the load approximates an infinite value, and attains its peak when the load is effectively zero. Finally, figure 6 shows the current distribution for different wire-lengths. It is evident that a short wire can be simulated by a very large continuous load from the end of the wire to the bottom of the cavity.

Discussions and Conclusions

The analysis of a rectangular cavity is always associated with a tripply infinite sum. The expressions describing the cavity behavior have been reduced from a tripply infinite sum to a doubly infinite sum by using the Fourier series representation. This essentially bypasses the Green's function approach and makes the solution easier to handle. The resulting doubly infinite sum is subsequently simplified to a singly infinite sum. The number of Fourier coefficients responsible for convergence is the key issue to this formulation since it determines the size of matrix to be in-

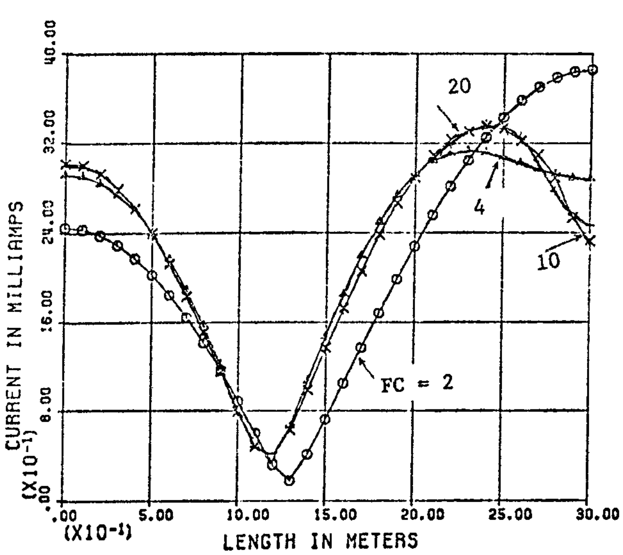

Fig. 2 Currents for various Fourier coefficients

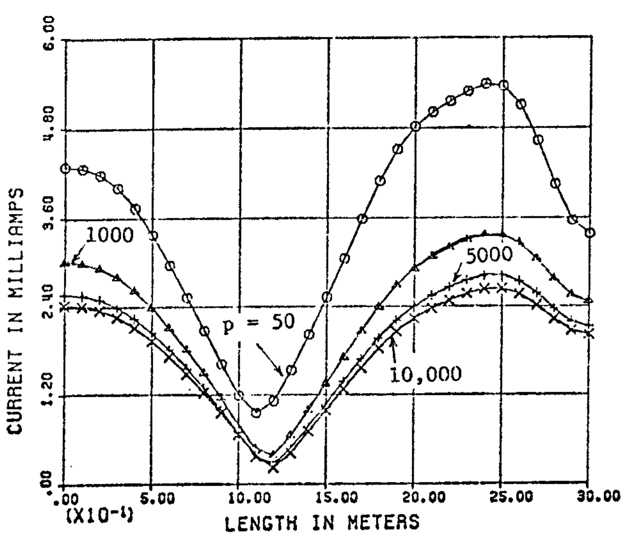

Fig. 3 Currents for various mode indices .

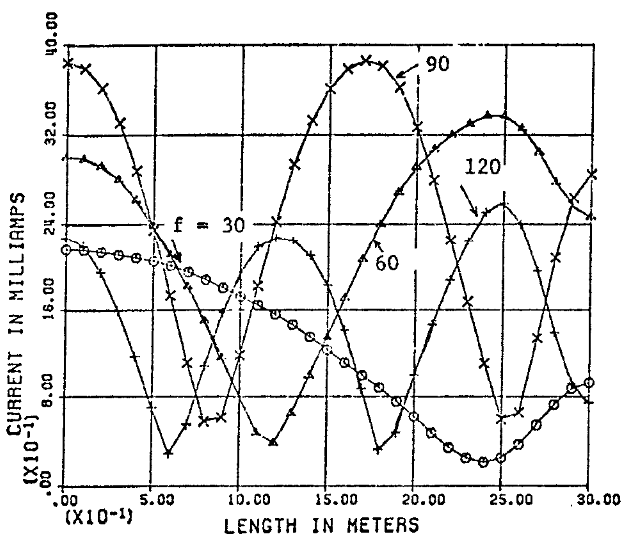

Fig. 4 Currents for various frequencies .

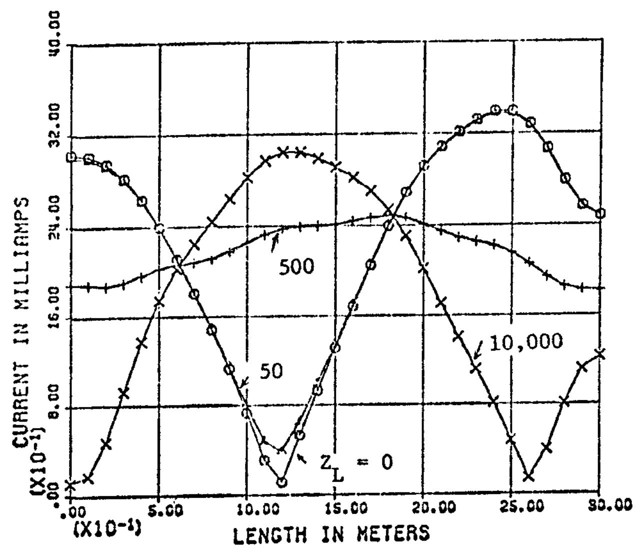

Fig. 5 Currents for various load conditions .

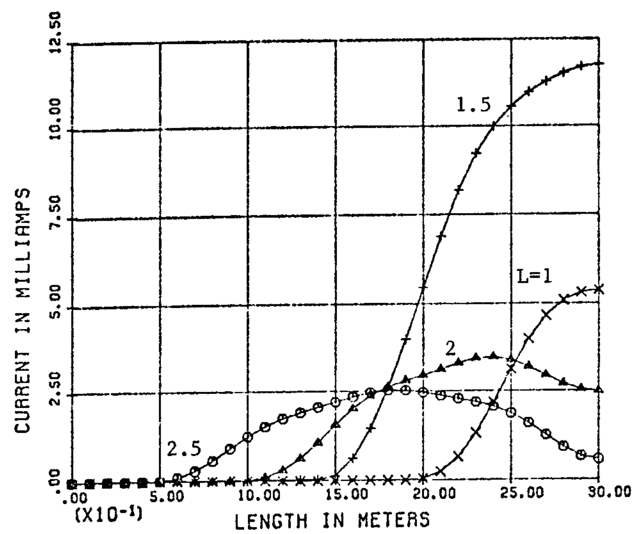

Fig. 6 Currents for different wire lengths .

verted during the process of computations.

This method of analysis is highly efficient and versatile. The tedius task of ordering the modes existing within the cavity has been eliminated in this formulation. The convergence of the infinite series is very fast as compared to other methods. This formulation can easily handle any load conditions at any point on the wire of any length.

References

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